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LETTER TO THE EDITOR

Critical polarization of the eight-vertex model

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Abstract. We investigate by series analysis the polarization of the eight-vertex model as a function of electric field at T_c and zero magnetic field, obtaining estimates of the exponent δ_e. These estimates are combined with a conjectured scaling relation to give a simple prediction for the spontaneous polarization exponent β_e.

The eight-vertex model has been extensively studied in recent years, there now being a number of critical exponents known exactly (Barber and Baxter 1973). A point of particular interest is that many of these indices vary continuously with the interaction parameters. The eight-vertex model was originally proposed in terms of dipole 'arrows' on a square lattice but can, as shown by Kadanoff and Wegner (1971) and Wu (1971), be regarded as an Ising spin system. As emphasized by Barber and Baxter, the two formulations lead naturally to two distinct types of ordering, a spin ordering with conjugate 'magnetic' field, $\mathscr{H}$, and an 'arrow' or 'spin-pair' ordering with conjugate electric field, J.

To refer to critical exponents in a system with two fields, we use the notation applied by Enting (1973) to the special case of the modified F model (Brascamp *et al* 1973). The exponents, β, γ', γ, δ are given subscripts e or m if they describe variations with electric or magnetic field respectively. Barber and Baxter have investigated β_m. The present work considers δ_e.

The hamiltonian considered is

$$H = -\sum_{\{i,j\}}^{(1)} J\sigma_i\sigma_j - \sum_{\{i,j\}}^{(2)} J_2\sigma_i\sigma_j - \sum_{\{i,j,k,l\}}^{(4)} J_4\sigma_i\sigma_j\sigma_k\sigma_l - \sum_{\{i\}} \mathscr{H}\sigma_i \tag{1}$$

where $\Sigma^{(1)}$, $\Sigma^{(2)}$, $\Sigma^{(4)}$ are respectively sums over all nearest-neighbour pairs, all second-neighbour pairs and all clusters of four spins on a square of nearest-neighbour bonds. The general eight-vertex model solved by Baxter (1972) allows the J_2 interaction to depend on direction but does not include the fields J, $\mathscr{H}$.

For $\mathscr{H}=0$ we have obtained power series expansions for the configurational free energy

$$\begin{aligned}\ln\Lambda &= \sum_{n=2}^{\infty} L_n(v,w)u^n \\ &= u^2(v^2w^4)+u^3(2v^4w^4)+u^4(2v^3w^6-4\tfrac{1}{2}v^4w^8+4v^5w^6+3v^6w^4) \\ &\quad +u^5(8v^5w^6-16v^6w^8+16v^7w^6+4v^8w^4)+u^6(6v^4w^8-28v^5w^{10}+20v^6w^8 \\ &\quad +33\tfrac{1}{3}v^6w^{12}+20v^7w^6-44v^7w^{10}-15v^8w^8+40v^9w^6+5v^{10}w^4)+u^7(36v^6w^8 \\ &\quad -160v^7w^{10}+120v^8w^8+188v^8w^{12}+40v^9w^6-240v^9w^{10}+60v^{10}w^8\end{aligned}$$

$$+80v^{11}w^{6}+6v^{12}w^{4})+u^{8}(1v^{4}w^{8}+18v^{5}w^{10}+4v^{6}w^{8}-150v^{6}w^{12}+88v^{7}w^{10}$$
$$+370v^{7}w^{14}+132v^{8}w^{8}-452v^{8}w^{12}-302\tfrac{1}{4}v^{8}w^{16}-376v^{9}w^{10}+556v^{9}w^{14}$$
$$+424v^{10}w^{8}+275v^{10}w^{12}+70v^{11}w^{6}-648v^{11}w^{10}+333\tfrac{1}{2}v^{12}w^{8}+140v^{13}w^{6}$$
$$+7v^{14}w^{4})+\ldots \tag{2}$$

with

$$u = \mathrm{e}^{-4\beta J}$$
$$v = \mathrm{e}^{-4\beta J_2}$$
$$w = \mathrm{e}^{-2\beta J_4}.$$

The polarization per bond is then

$$P = 1+\tfrac{1}{2}\frac{\partial}{\partial J}kT\ln\Lambda. \tag{3}$$

The exponent δ_{e} is defined by

$$P(u) \sim (1-u)^{1/\delta_{\mathrm{e}}}, \qquad (T = T_{\mathrm{c}}).$$

We have used Baxter's formula for T_{c} and obtained estimates of $1/\delta_{\mathrm{e}}$ from Padé approximants to $(1-u)$ $(\mathrm{d}/\mathrm{d}u)$ ln P evaluated at $u=1$. These estimates are shown in figure 1.

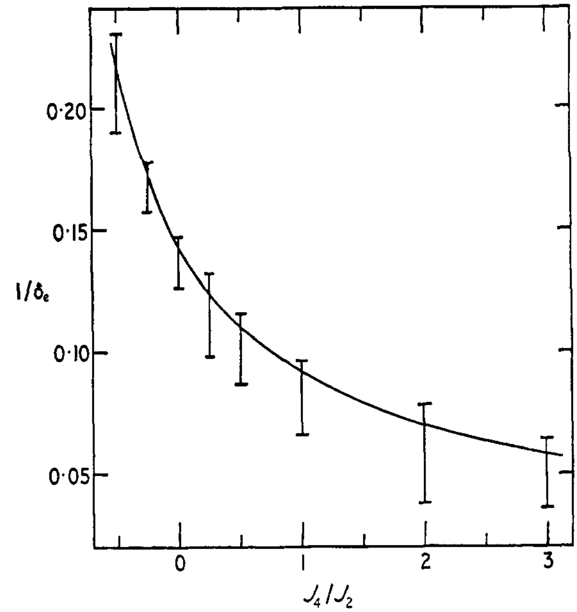

Figure 1. The conjectured variation $1/\delta_{\mathrm{e}}$ with J_4/J_2. Numerical estimates are represented by the error bars.

To interpret these results we postulate a scaling relation in the form of a generalized homogeneous function (GHF)

$$\lambda G(\epsilon, \mathscr{H}, J) = G(\lambda^{a}\epsilon, \lambda^{b}\mathscr{H}, \lambda^{c}J) \tag{4}$$

where $\epsilon = T-T_{\mathrm{c}}$. The consequences of this type of assumption have been discussed by Hankey and Stanley (1972). For $J = 0$ the work of Barber and Baxter suggests

$$a = \frac{\bar{\mu}}{\pi} = 1-\frac{1}{\pi}\cos^{-1}\left(\tanh\frac{2J_4}{kT_{\mathrm{c}}}\right), \qquad \text{range 0 to 1} \tag{5}$$

$$b = \tfrac{15}{16}. \tag{6}$$

This predicts $\delta_m = 15$ which is consistent with series results obtained by Gaunt (1974).

For $J_4 = 0$, ie $a = \frac{1}{2}$ the GHF hypothesis is supported by the work of Grover (1973) which together with the work of Brascamp *et al* (1973) and Enting (1973, 1974) indicates

$$c = \tfrac{7}{8} \tag{7}$$

at $J_4 = 0$. The scaling-perturbation theory approach of Kadanoff and Wegner (1971) shows that c varies with J_4 and if we assume that β_e varies linearly with $\pi/\bar{\mu}$ (as γ_m, β_m are conjectured to do; Barber and Baxter 1973), the value and derivative of β_e at $J_4 = 0$ give

$$\beta_e = \tfrac{1}{4}\left(\frac{\pi}{\bar{\mu}}\right) - \tfrac{1}{4} \qquad \text{range 0 to } \infty \tag{8}$$

$$c = \tfrac{3}{4} + \tfrac{1}{4}\frac{\bar{\mu}}{\pi} \qquad \text{range } \tfrac{3}{4} \text{ to } 1 \tag{9}$$

$$\delta_e = \frac{c}{1-c} \qquad \text{range 3 to } \infty. \tag{10}$$

(The range of values taken by a and c is consistent with the restrictions given by Hankey and Stanley.) The values of $1/\delta_e$ obtained from these expressions are also shown in figure 1 for comparison with the series estimates. The agreement is fairly good and it seems that (9), (10) give a good representation of δ_e over the range considered.

In conclusion we repeat that the general eight-vertex model includes anisotropic second-neighbour interactions. Barber and Baxter (1973) found that a single variable $\bar{\mu}$ was sufficient to describe α, β_m for arbitrary anisotropy. Whether the same variable can also describe δ_e in the anisotropic case remains an open question.

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